

ON QUESTIONS WHICH ARE CONNECTED WITH TALAGRAND PROBLEM

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ABSTRACT. We prove the following results.

1. If X is a α -favourable space, Y is a regular space, in which every separable closed set is compact, and $f : X \times Y \rightarrow \mathbb{R}$ is a separately continuous everywhere jointly discontinuous function, then there exists a subspace $Y_0 \subseteq Y$ which is homeomorphic to $\beta\mathbb{N}$.

2. There exist a α -favourable space X , a dense in $\beta\mathbb{N} \setminus \mathbb{N}$ countably compact space Y and a separately continuous everywhere jointly discontinuous function $f : X \times Y \rightarrow \mathbb{R}$.

Besides, it was obtained some conditions equivalent to the fact that the space $C_p(\beta\mathbb{N} \setminus \mathbb{N}, \{0, 1\})$ of all continuous functions $x : \beta\mathbb{N} \setminus \mathbb{N} \rightarrow \{0, 1\}$ with the topology of point-wise convergence is a Baire space.

1. INTRODUCTION

Investigation of joint continuity points set of separately continuous functions of two variables was started by R. Baire in [1]. It was continued in papers of many mathematicians (H. Hahn, W. Serpinski, V. Moran, I. Namioka, M. Talagrand, W. Rudin, V. Maslyuchenko and other; see, for example, [2] and the literature given there). I. Namioka shows in [3] that for every strongly countably complete space X , compact space Y and separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ there exists a dense in X G_δ -set $A \subseteq X$ such that f is jointly continuous at every point of set $A \times Y$. This result intensified the investigation of separately continuous functions defined on the product of Baire and compact spaces. In particular, it was constructed in [4] an example of α -favorable space X , compact space Y and separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ such that the projection on X of the set $D(f)$ of discontinuity points set of f coincides with X . In this connection the following question was formulated in [4, Problem 3].

Problem 1.1. *Let X be a Baire space, Y be a compact space and $f : X \times Y \rightarrow \mathbb{R}$ be a separately continuous function. Is the function f continuous at least at one point?*

It was shown in [5] that this question has the negative answer if the compactness of Y to replace by τ -compactness, where τ is an arbitrary infinite cardinal (a topological space X is called τ -compact, if every open cover of X with the cardinality $\leq \tau$ has a finite subcover).

Note that for a completely regular space Y and the space $X = C_p(Y, [0, 1])$ of all continuous functions $x : Y \rightarrow [0, 1]$ with the topology of pointwise convergence, or

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for a Hausdorff space Y with a open-closed base and the space $X = C_p(Y, \{0, 1\})$ of all continuous functions $x : Y \rightarrow \{0, 1\}$ with the topology of pointwise convergence the separately continuous function $f : X \times Y \rightarrow \mathbb{R}$, $f(x, y) = x(y)$, is everywhere discontinuous. Therefore, it was naturally arises in the connection with Talagrand's Problem 1.1 the question on investigation Baire property of spaces $C_p(Y, [0, 1])$ and $C_p(Y, \{0, 1\})$ for Hausdorff compact spaces Y .

In this paper we investigate the problem on the existence of everywhere discontinuous separately continuous function defined on the product of an α -favorable space X and a space Y , which satisfies a compactness-type conditions. Firstly we show that for an α -favorable space X and a regular space Y , in which every separable closed set is compact, the existence of an everywhere discontinuous function $f : X \times Y \rightarrow \mathbb{R}$, which quasicontinuous with respect to the first variable and continuous with respect to the second variable, imply the existence a subspace of Y which is homeomorphic to Stone-Cech compactification $\beta\mathbb{N}$ of countable discrete space \mathbb{N} . Further, we construct an example of everywhere discontinuous separately continuous function defined on the product of an α -favorable space X and countably compact subspace Y of space $\beta\mathbb{N} \setminus \mathbb{N}$. In the finish section we obtain some equivalent reformulations of the Baire property of the space of all continuous functions $x : \beta\mathbb{N} \setminus \mathbb{N} \rightarrow \{0, 1\}$ with the topology of pointwise convergence.

2. EVERYWHERE DISCONTINUOUS KC -FUNCTIONS

Let X, Y, Z be topological spaces and $f : X \times Y \rightarrow Z$. For every $x_0 \in X$ and $y_0 \in Y$ the mappings $f^{x_0} : Y \rightarrow Z$ $f_{y_0} : X \rightarrow Z$ are defined by:

$$f^{x_0}(y) = f(x_0, y) \quad \text{and} \quad f_{y_0}(x) = f(x, y_0)$$

for every $x \in X$ and $y \in Y$.

A mapping $f : X \rightarrow Y$ defined on a topological space X and valued in a topological space Y is called *quasicontinuous at a point* $x_0 \in X$, if for every neighborhoods U of x_0 in X and V of $f(x_0)$ in Y there exists an open in X nonempty set $U_1 \subseteq U$ such that $f(U_1) \subseteq V$. A mapping $f : X \rightarrow Y$ which is quasicontinuous at every point $x \in X$ is called *quasicontinuous*.

For topological spaces X, Y and Z the set of all mappings $f : X \times Y \rightarrow Z$ which is quasicontinuous with respect to the first variable and continuous with respect to the second variable we denote by $KC(X \times Y, Z)$.

Lemma 2.1. *Let X, Y, Z be topological spaces, $f \in KC(X \times Y, Z)$, W_0, W_1 open in Z nonempty sets such that $\overline{f^{-1}(W_0)} = \overline{f^{-1}(W_1)} = X \times Y$. Then for every $n \in \mathbb{N}$, open in X nonempty sets G_1, G_2, \dots, G_n and reals $\theta_1, \theta_2, \dots, \theta_n \in \{0, 1\}$ there exists $y_0 \in Y$, open in X nonempty sets U_1, U_2, \dots, U_n such that $U_k \subseteq G_k$ $f_{y_0}(U_k) \subseteq W_{\theta_k}$ for every $1 \leq k \leq n$.*

Proof. Since all sets $f^{-1}(W_{\theta_k})$ are dense in $X \times Y$, for every $k \leq n$ the set $B_k = \{y \in Y : f(G_k \times \{y\}) \cap W_{\theta_k} \neq \emptyset\}$ is dense in Y . Moreover, the continuity of f with respect to the second variable imply that all sets B_k are open in Y . Therefore, the set $\bigcap_{k=1}^n B_k$ is nonempty. We take $y_0 \in \bigcap_{k=1}^n B_k$. There exist points $x_k \in G_k$ for $k \leq n$ such that $f(x_k, y_0) \in W_{\theta_k}$. Now using the quasicontinuity of f with respect to the first variable we found nonempty open in X sets $U_k \subseteq G_k$ such that $f_{y_0}(U_k) \subseteq W_{\theta_k}$ for every $k \leq n$. \square

Let X be a topological space. Define the Shoquet game on X in which two players α and β participate. A nonempty open in X set U_0 is the first move of β and a nonempty open in X set $V_1 \subseteq U_0$ is the first move of α . Further β chooses a nonempty open in X set $U_1 \subseteq V_1$ and α chooses a nonempty open in X set $V_2 \subseteq U_1$ and so on. The player α wins if $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$. Otherwise β wins.

A topological space X is called α -favorable if α has a winning strategy in this game. A topological space X is called β -unfavorable if β has no winning strategy in this game. Clearly, any α -favorable topological space X is a β -unfavorable space. It was shown in [6] that a topological game X is Baire if and only if X is β -unfavorable.

Let X be a topological space, $x_0 \in X$, \mathcal{U} be a system of all neighborhoods of x_0 in X and $f : X \rightarrow \mathbb{R}$. The real

$$\omega_f(x_0) = \inf_{U \in \mathcal{U}} \sup_{x', x'' \in U} |f(x') - f(x'')|$$

is called by *the oscillation of the function f at the point x_0* .

Theorem 2.2. *Let X be an α -favorable space, Y be a Baire space and $f \in KC(X \times Y, \mathbb{R})$ such that $D(f) = X \times Y$. Then there exists a sequence $(y_n)_{n=1}^{\infty}$ of points $y_n \in Y$ such that for every set $N \in \mathbb{N}$ there exists a continuous function $g : Y \rightarrow [0, 1]$ such that $g(y_n) = 1$, if $n \in N$, and $g(y_n) = 0$, if $n \in \mathbb{N} \setminus N$.*

Proof. According to [6], the space $X \times Y$ is Baire. Therefore there exist open in X and Y respectively sets $X_1 \subseteq X$ and $Y_1 \subseteq Y$, and $\varepsilon > 0$ such that $\omega_f(x, y) \geq \varepsilon$ for every $(x, y) \in X_1 \times Y_1$. Using the fact that $X_1 \times Y_1$ is Baire, we found nonempty open in X and Y respectively sets $X_0 \subseteq X_1$ and $Y_0 \subseteq Y_1$, reals $a, b \in \mathbb{R}$ with $a < b$ such that the sets $f^{-1}(W_0)$ and $f^{-1}(W_1)$ are dense in $X_0 \times Y_0$, where $W_0 = (-\infty, a)$ and $W_1 = (b, +\infty)$.

Let \mathcal{T} is the topology of the space X and $\tau : \bigcup_{n=1}^{\infty} \mathcal{T}^{2n+1} \rightarrow \mathcal{T}$ is a winning strategy of α in the Shoquet game on the topological space X .

For every $n \in \mathbb{N} \cup \{\omega_0\}$, $\xi = (\xi_1, \xi_2, \dots) \in \{0, 1\}^n$ and $k < n$ we put $\xi|_k = (\xi_1, \xi_2, \dots, \xi_k)$.

Using the induction on $n \in \mathbb{N}$ we construct sequences of families $(U_\xi : \xi \in \{0, 1\}^n)$ and $(V_\xi : \xi \in \{0, 1\}^n)$ of open in X nonempty sets U_ξ and V_ξ and a sequence $(y_n)_{n=1}^{\infty}$ of points $y_n \in Y$ such that:

- (i) $V_\xi = \tau(U_{\xi|_1}, V_{\xi|_1}, \dots, U_\xi)$ for every $n \in \mathbb{N}$ $\xi \in \{0, 1\}^n$;
- (ii) $U_\xi \subseteq V_{\xi|_n}$ for every $n \in \mathbb{N}$ and $\xi \in \{0, 1\}^{n+1}$;
- (iii) $f_{y_n}(U_\xi) \subseteq W_{\xi_n}$ for every $n \in \mathbb{N}$ $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \{0, 1\}^n$.

According to Lemma 2.1, we choose a point $y_1 \in Y_0$ and open in X nonempty sets U_0 and U_1 such that $f_{y_1}(U_\xi) \subseteq W_\xi$ for every $\xi \in \{0, 1\}$. Put $V_0 = \tau(U_0)$ and $V_1 = \tau(U_1)$.

Assume that the points $y_k \in Y$, the families $(U_\xi : \xi \in \{0, 1\}^k)$ and $(V_\xi : \xi \in \{0, 1\}^k)$ for $k \leq n$ are constructed. For every $\xi = (\xi_1, \xi_2, \dots, \xi_{n+1}) \in \{0, 1\}^{n+1}$ put $G_\xi = V_{\xi|_n}$ $\theta_\xi = \xi_{n+1}$. Then according to Lemma 2.1, there exist $y_{n+1} \in Y$ and a family $(U_\xi : \xi \in \{0, 1\}^{n+1})$ of nonempty open in X sets U_ξ such that $U_\xi \subseteq G_\xi$ and $f_{y_{n+1}}(U_\xi) \subseteq W_{\theta_\xi}$, that is the conditions (ii) and (iii) are true for every $\xi \in \{0, 1\}^{n+1}$. It remains to put $V_\xi = \tau(U_{\xi|_1}, V_{\xi|_1}, \dots, U_\xi)$ for all $\xi \in \{0, 1\}^{n+1}$.

Show that the sequence $(y_n)_{n=1}^{\infty}$ is the required. Let $N \subseteq \mathbb{N}$. Put $\xi_n = 1$, if $n \in N$, $\xi_n = 0$, if $n \in \mathbb{N} \setminus N$, and $\xi = (\xi_n)_{n=1}^{\infty}$. According to (i) and (ii), we have

$U_{\xi|_{n+1}} \subseteq V_{\xi|_n} \subseteq U_{\xi|_n}$ for every $n \in \mathbb{N}$. Note that the player α plays accordingly with the winner strategy τ in the Shoquet game

$$U_{\xi|_1} \subseteq V_{\xi|_1} \subseteq \dots$$

Therefore $\bigcap_{n=1}^{\infty} U_{\xi|_n} \neq \emptyset$.

Let $x_0 \in \bigcap_{n=1}^{\infty} U_{\xi|_n}$. According to (iii), we have $f(x_0, y_n) \in W_1$, if $n \in N$, and $f(x_0, y_n) \in W_0$, $n \in \mathbb{N} \setminus N$. Take an continuous function $\varphi : \mathbb{R} \rightarrow [0, 1]$ such that $W_0 \subseteq \varphi^{-1}(0)$ and $W_1 \subseteq \varphi^{-1}(1)$. Then the continuous function $g : Y \rightarrow [0, 1]$, $g(y) = \varphi(f(x_0, y))$, is the required. \square

The following Corollary is a main result of this section.

Corollary 2.3. *Let X be an α -favorable space, Y be a regular space in which every separable closed set is compact and $f \in KC(X \times Y, \mathbb{R})$ such that $D(f) = X \times Y$. Then there exists a compact in Y set Y_0 , which is homeomorphic to the space $\beta\mathbb{N}$.*

Proof. It easy to see that every regular space, in which each separable closed set is compact, is α -favorable, in particular, a Baire space. According to Theorem 2.2, we choose a sequence $(y_n)_{n=1}^{\infty}$ which satisfies the corresponding condition and put $Y_0 = \overline{\{y_n : n \in \mathbb{N}\}}$. Then according to [7, Corollary 3.6.4] the space Y_0 is homeomorphic to $\beta\mathbb{N}$. \square

3. STONE-CECH COMPACTIFICATION AND p -SETS

A system \mathcal{A} of subsets of a set X is called *ultrafilter on X* , if the following conditions hold:

- (a) $\bigcap \mathcal{B} \neq \emptyset$ for every finite system $\mathcal{B} \subseteq \mathcal{A}$;
- (b) either $A \in \mathcal{A}$ or $X \setminus A \in \mathcal{A}$ for every set $A \subseteq X$.

Let \mathcal{F} be the collection of all ultrafilters on \mathbb{N} . Clearly (see [7, Corollary 3.6.4]) that a mapping $\varphi : \beta\mathbb{N} \rightarrow \mathcal{F}$, $\varphi(x) = \{A \subseteq \mathbb{N} : x \in \overline{A}\}$, is a bijection, besides $\varphi(n) = \{A \subseteq \mathbb{N} : n \in A\}$ for every $n \in \mathbb{N}$. Moreover, for every $x \in \beta\mathbb{N} \setminus \mathbb{N}$ the ultrafilter $\varphi(x)$ is called *nontrivial* and it has the following property: if $A \in \varphi(x)$ and $B \subseteq \mathbb{N}$ such that $|A \setminus B| < \aleph_0$ then $B \in \varphi(x)$.

Further, the elements $x \in \beta\mathbb{N} \setminus \mathbb{N}$ we will identify with $\varphi(x)$. Note that for every closed-open nonempty set $U \subseteq \beta\mathbb{N} \setminus \mathbb{N}$ there exists an infinite set $A \subseteq \mathbb{N}$ such that $U = \{x \in \beta\mathbb{N} \setminus \mathbb{N} : A \in x\}$.

Lemma 3.1. *Let $X = \beta\mathbb{N} \setminus \mathbb{N}$, $(A_n)_{n=1}^{\infty}$ $(B_n)_{n=1}^{\infty}$ be sequences of closed in X sets $A_n, B_n \subseteq X$ such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, where $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then $\overline{A} \cap \overline{B} = \emptyset$.*

Proof. Using the induction on n it easy to construct sequences $(U_n)_{n=1}^{\infty}$ and $(V_n)_{n=1}^{\infty}$ of closed-open in X sets U_n and V_n such that $A_n \subseteq U_n$, $B_n \subseteq V_n$ for every $n \in \mathbb{N}$ and $(\bigcup_{n=1}^{\infty} U_n) \cap (\bigcup_{n=1}^{\infty} V_n) = \emptyset$. We choose sequences $(S_n)_{n=1}^{\infty}$ and $(T_n)_{n=1}^{\infty}$ of sets $S_n, T_n \subseteq \mathbb{N}$ such that $U_n = \{x \in X : S_n \in x\}$ and $V_n = \{x \in X : T_n \in x\}$ for every $n \in \mathbb{N}$. Since $U_n \cap V_m = \emptyset$, $|S_n \cap T_m| < \aleph_0$ for every $n, m \in \mathbb{N}$. Put

$$S = \bigcup_{n=1}^{\infty} (S_n \setminus (\bigcup_{k=1}^n T_k)) \text{ and } T = \bigcup_{n=1}^{\infty} (T_n \setminus (\bigcup_{k=1}^n S_k)).$$

We show that $S \cap T = \emptyset$. Suppose that $m \in S \cap T$. Taking into account that $S \subseteq \bigcup_{n=1}^{\infty} S_n$ and $T \subseteq \bigcup_{n=1}^{\infty} T_n$, we put $i = \min\{n \in \mathbb{N} : m \in S_n\}$ $j = \min\{n \in \mathbb{N} : m \in T_n\}$.

If $i \leq j$, then $m \notin T_n$ for $n < j$ and $m \notin T_n \setminus (\bigcup_{k=1}^n S_k)$ for $n \geq j$. Thus, $m \notin T$, a contradiction. Analogously, $m \notin S$ if $j \leq i$.

Moreover, note that $S_n \setminus S \subseteq S_n \setminus (S_n \setminus \bigcup_{k=1}^n T_k) \subseteq \bigcup_{k=1}^n (S_n \cap T_k)$ and $T_n \setminus T \subseteq T_n \setminus (T_n \setminus \bigcup_{k=1}^n S_k) \subseteq \bigcup_{k=1}^n (T_n \cap S_k)$ for every $n \in \mathbb{N}$. Therefore all sets $S_n \setminus S$ and $T_n \setminus T$ are finite, $U_n \subseteq U = \{x \in X : S \in x\}$ and $V_n \subseteq V = \{x \in X : T \in x\}$ for every $n \in \mathbb{N}$, besides the closed-open in X sets U and V such that $U \cap V = \emptyset$. \square

The next result follows from [7, Corollary 3.6.4].

Corollary 3.2. *Let $A \subseteq \beta\mathbb{N} \setminus \mathbb{N}$ be a countable set. Then the closure \overline{A} of A in the space $\beta\mathbb{N} \setminus \mathbb{N}$ is homeomorphic to the Stone-Cech compactification of the space A .*

A subset A of a topological space X is called p -set, if

$$A \subseteq G = \text{int} \left(\bigcap_{n=1}^{\infty} G_n \right)$$

for every sequence $(G_n)_{n=1}^{\infty}$ of open in X sets G_n with $A \subseteq G_n$ for every $n \in \mathbb{N}$.

Proposition 3.3. *Let \mathcal{P} be a system of all closed nowhere dense p -sets in $X = \beta\mathbb{N} \setminus \mathbb{N}$. Then*

- (i) *the set $\bigcup_{P \in \mathcal{P}} P$ is dense in X ;*
- (ii) *$U \cap P \in \mathcal{P}$ for every closed-open in X sets U and $P \in \mathcal{P}$;*
- (iii) *$P = \bigcup_{n=1}^{\infty} P_n \in \mathcal{P}$ for every sequence $(P_n)_{n=1}^{\infty}$ of sets $P_n \in \mathcal{P}$.*

Proof. Conditions (i) and (ii) immediately follows from [7, exercise 3.6.]. We prove (iii). Let $P_n \in \mathcal{P}$ for every $n \in \mathbb{N}$. Since every nonempty G_δ -set in X has nonempty interior (see [7, exercise 3.6.]), the set $P = \bigcup_{n=1}^{\infty} P_n$ is nowhere dense in X . It remains to show that P is a p -set in X .

Let $(G_n)_{n=1}^{\infty}$ be a sequence of open in X sets G_n such that $P \subseteq \bigcap_{n=1}^{\infty} G_n$. Put $A_n = X \setminus G_n$ for every $n \in \mathbb{N}$, $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} P_n$. Since $P_n \in \mathcal{P}$ for every $n \in \mathbb{N}$, $B \subseteq \text{int}(\bigcap_{n=1}^{\infty} G_n)$, that is $B \cap \overline{A} = \emptyset$. Moreover, $P = \overline{B} \subseteq \bigcap_{n=1}^{\infty} G_n$, therefore $\overline{B} \cap A = \emptyset$. According to Lemma 3.1, we have $\overline{A} \cap \overline{B} = \emptyset$, that is $P \subseteq \text{int}(\bigcap_{n=1}^{\infty} G_n)$. \square

Now we give an example of everywhere discontinuous separately continuous function defined on the product of α -favorable space X and countably compact dense subspace of $\beta\mathbb{N} \setminus \mathbb{N}$.

Example 3.4. Let X be a set of all continuous functions $x : \beta\mathbb{N} \setminus \mathbb{N} \rightarrow \{0, 1\}$, \mathcal{P} be a system of all closed nowhere dense p -sets $P \subseteq \beta\mathbb{N} \setminus \mathbb{N}$ and $Y = \bigcup_{P \in \mathcal{P}} P$.

We consider the space X with the topology of uniform convergence on sets of the system \mathcal{P} . That is for every $x \in X$ the system $\{U(x, P) : P \in \mathcal{P}\}$ forming a base of neighborhoods of x in the space X , where $U(x, P) = \{x' \in X : x'(t) = x(t) \ \forall t \in P\}$.

Consider the separately continuous function $f : X \times Y \rightarrow \mathbb{R}$, $f(x, y) = x(y)$. Since in Y the system of all closed-open sets forming a base of the topology and every set $P \in \mathcal{P}$ is nowhere dense in Y , the function f is discontinuous at every point $(x_0, y_0) \in X \times Y$.

Now we show that the space X is α -favorable. Let $(U_n)_{n=1}^\infty$ is a decreasing sequence of nonempty basic open sets in X . Then there exist increasing sequences $(P_n)_{n=1}^\infty$ and $(Q_n)_{n=1}^\infty$ of sets $P_n, Q_n \in \mathcal{P}$ such that

$$U_n = \{x \in X : x(y) = 0 \ \forall y \in P_n \text{ and } x(y) = 1 \ \forall y \in Q_n\}.$$

Put $P = \overline{\bigcup_{n=1}^\infty P_n}$ and $Q = \overline{\bigcup_{n=1}^\infty Q_n}$. Proposition 3.3 imply that $P, Q \in \mathcal{P}$. Moreover, it follows from the definition of p -set that $P_n \cap Q = P \cap Q_n = \emptyset$ for every $n \in \mathbb{N}$. Therefore according to Lemma 3.1, $P \cap Q = \emptyset$. Now choose a continuous on $\beta\mathbb{N} \setminus \mathbb{N}$ function x_0 such that $x_0(y) = 0$ for every $y \in P$ and $x_0(y) = 1$ for every $y \in Q$ and obtain $x_0 \in \bigcap_{n=1}^\infty U_n$.

A positive answer to the following question gives the solution of Talagrand problem.

Question 3.5. Is there equality $\beta\mathbb{N} \setminus \mathbb{N} = \bigcup_{P \in \mathcal{P}} P$, where \mathcal{P} is the system of all closed nowhere dense p -sets in $\beta\mathbb{N} \setminus \mathbb{N}$?

4. SOME PROPERTY OF $C_p(\beta\mathbb{N} \setminus \mathbb{N}, \{0, 1\})$

Let X be topological space and $(A_n)_{n=1}^\infty$ be a sequence of sets $A_n \subseteq X$. We say that the sequence $(A_n)_{n=1}^\infty$ weakly converges to $x_0 \in X$ in X , if for every neighborhood U of x_0 in X there exists an integer $n_0 \in \mathbb{N}$ such that $U \cap A_n \neq \emptyset$ for every $n \geq n_0$.

Theorem 4.1. Let $Y = \beta\mathbb{N} \setminus \mathbb{N}$ and $X = C_p(Y, \{0, 1\})$. Then the following conditions are equivalent:

- (i) X is meagre;
- (ii) X is not Baire;
- (iii) there exists a sequence $(E_n)_{n=1}^\infty$ of finite pairwise disjoint sets $E_n \subseteq Y$ which weakly converges to a point $y_0 \in Y$;
- (iv) there exists a sequence $(E_n)_{n=1}^\infty$ of finite pairwise disjoint sets $E_n \subseteq Y$ which weakly converges to every point $y \in \bigcup_{n=1}^\infty E_n$.

Proof. For every disjoint sets $A, B \subseteq Y$ we put

$$U(A, B) = \{x \in X : x(a) = 0 \ \forall a \in A, x(b) = 1 \ \forall b \in B\}.$$

Clearly that the system

$$\{U(A, B) : A, B \subseteq Y \text{ are finite and disjoint}\}$$

forming a base of the topology of X .

The implications (i) \Rightarrow (ii) and (iv) \Rightarrow (iii) are obvious.

(ii) \Rightarrow (iii). Let $A_0, B_0 \subseteq Y$ are finite disjoint sets such that $X_0 = U(A_0, B_0)$ is meagre in X , that is $X_0 = \bigcup_{n=1}^{\infty} X_n$, where $(X_n)_{n=1}^{\infty}$ is a increasing sequence of nowhere dense in X sets.

Lemma 4.2. *For every $n \in \mathbb{N}$ and finite set $C \subseteq Y$ there exist finite disjoint sets $A, B \subseteq Y \setminus C$ such that $U(A, B) \cap X_n = \emptyset$.*

Proof. Let $D = C \setminus (A_0 \cup B_0) = \{d_1, \dots, d_m\}$, moreover without loss of generality we can propose that $m \geq 1$. Let D_1, \dots, D_{2^m} are all subsets of set D . We put $C_k = D \setminus D_k$ for $k = 1, \dots, 2^m$.

Show that $X_0 = \bigcup_{k=1}^{2^m} U(A_0 \cup C_k, B_0 \cup D_k)$. Since $U(A_0 \cup C_k, B_0 \cup D_k) \subseteq X_0$ for every $k = 1, \dots, 2^m$, $\bigcup_{k=1}^{2^m} U(A_0 \cup C_k, B_0 \cup D_k) \subseteq X_0$.

Let $x \in X_0$. Using $k \in \{1, \dots, 2^m\}$ such that $C_k = \{y \in D : x(y) = 0\}$ $D_k = \{y \in D : x(y) = 1\}$ we obtain that $x \in U(A_0 \cup C_k, B_0 \cup D_k)$.

Since X_n is meagre in X , there exist finite disjoint sets $S_1, T_1 \subseteq Y \setminus (A_0 \cup B_0 \cup D)$ such that $U(A_0 \cup C_1 \cup S_1, B_0 \cup D_1 \cup T_1) \cap X_n = \emptyset$. Further, using the fact that X_n is meagre in X by the induction on k we construct sequences $(S_k)_{k=1}^{2^m}$ and $(T_k)_{k=1}^{2^m}$ of pairwise disjoint sets $S_k, T_k \subseteq Y$ such that $(S_k \cup T_k) \cap \left(\bigcup_{i=1}^{k-1} (S_i \cup T_i) \cup A_0 \cup B_0 \cup D \right) = \emptyset$ and $U\left(\bigcup_{i=1}^k S_i \cup A_0 \cup C_k, \bigcup_{i=1}^k T_i \cup B_0 \cup D_k\right) \cap X_n = \emptyset$ for every $k \in \{1, \dots, 2^m\}$.

We put $A = \bigcup_{k=1}^{2^m} S_k$ and $B = \bigcup_{k=1}^{2^m} T_k$. Show that $U(A, B) \cap X_n = \emptyset$. Assume that $x \in U(A, B) \cap X_n$. Since $X_n \subseteq X_0$, there exists $k \in \{1, \dots, 2^m\}$ such that $x \in U(A_0 \cup C_k, B_0 \cup D_k)$. Then $x \in U(A \cup A_0 \cup C_k, B \cup B_0 \cup D_k) \cap X_n \subseteq U\left(\bigcup_{i=1}^k S_i \cup A_0 \cup C_k, \bigcup_{i=1}^k T_i \cup B_0 \cup D_k\right) \cap X_n$. But this contradicts to the choice of sets S_k and T_k . \square

It follows from Lemma 4.2 that there exist sequences $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ of finite disjoint sets $A_n, B_n \subseteq Y$ such that $(A_n \cup B_n) \cap \left(\bigcup_{k=0}^{n-1} (A_k \cup B_k) \right) = \emptyset$ and $U(A_n, B_n) \cap X_n = \emptyset$ for every $n \in \mathbb{N}$.

Suppose that (iii) is false. We consider the sequence $(E_n)_{n=1}^{\infty}$ of pairwise disjoint sets $E_n = A_n \cup B_n$. Using the denial of (i) and the finiteness of $E_0 = A_0 \cup B_0$ we found a finite set $N_1 \subseteq \mathbb{N}$ such that $E_0 \cap \bigcup_{n \in N_1} E_n = \emptyset$.

Using similar reasoning with respect to the set E_{n_1} , where $n_1 = \min N_1$, the sequence $(E_n)_{n \in N_1}$, we choose an infinite set $N_2 \subseteq N_1$ such that $E_{n_1} \cap \bigcup_{n \in N_2} E_n = \emptyset$.

Continuing this process to infinity we obtain a strictly decreasing sequence $(N_k)_{k=1}^\infty$ of infinite sets $N_k \subseteq \mathbb{N}$ such that

$$E_{n_{k-1}} \cap \overline{\bigcup_{n \in N_k} E_n} = \emptyset,$$

for every $k \in \mathbb{N}$, where $n_k = \min N_k$ and $n_0 = 0$.

Put $\tilde{A}_k = A_{n_{k-1}}$, $\tilde{B}_k = B_{n_{k-1}}$ for every $k \in \mathbb{N}$, $A = \bigcup_{k=1}^\infty \tilde{A}_k$ and $B = \bigcup_{k=1}^\infty \tilde{B}_k$.

According to the choice of $(n_k)_{k=1}^\infty$ we have $E_{n_k} \cap \left(\overline{\bigcup_{i \neq k} E_{n_i}} \right) = \emptyset$ for every $k \in \mathbb{N}$.

Therefore $\tilde{A}_k \cap \overline{B} = \overline{A} \cap \tilde{B}_k = \emptyset$ for every $k \in \mathbb{N}$ and $\overline{A} \cap \overline{B} = \emptyset$ according to Lemma 3.1. Hence, $U(A, B) \neq \emptyset$, that is there exists $x_0 \in U(A, B)$. Now since $A_0 \subseteq A$ and $B_0 \subseteq B$, $x_0 \in U(A_0, B_0) = X_0$. On other hand, using that $A_{n_k} \subseteq A$ and $B_{n_k} \subseteq B$ for every $k \in \mathbb{N}$, we obtain that

$$x_0 \in \bigcap_{k=1}^\infty U(A_{n_k}, B_{n_k}) \subseteq \bigcap_{k=1}^\infty (X \setminus X_{n_k}) = X \setminus \left(\bigcup_{k=1}^\infty X_{n_k} \right) = X \setminus X_0.$$

This gives a contradiction

(iii) \Rightarrow (iv). Let $(E_n)_{n=1}^\infty$ be a sequence of finite pairwise disjoint sets $E_n \subseteq Y$, which weakly converges to $y_0 \in Y$. Let

$$E = \bigcup_{n=1}^\infty E_n = \{y_n : n \in \mathbb{N}\}.$$

Using the induction on k it easy to construct a strictly decreasing sequence of infinite sets $N_k \subseteq \mathbb{N}$ such that for every $k \in \mathbb{N}$ at least one of the following conditions

(a) $y_k \notin \overline{\bigcup_{n \in N_k} E_n}$;

(b) the sequence $(E_n)_{n \in N_k}$ weakly converges to y_k ;

holds.

We take a strictly increasing sequence $(n_k)_{k=1}^\infty$ of integers $n_k \in N_k$. For every $k \in \mathbb{N}$ we put

$$A_k = \{y_m \in E_{n_k} : \text{sequence } (E_n)_{n \in N_m} \text{ weakly converges to } y_m\}.$$

We show that there exists an integer k_0 such that $A_k \neq \emptyset$ for every $k \geq k_0$.

Suppose that there exists an infinite set $M \subseteq \mathbb{N}$ such that $A_k = \emptyset$ for every $k \in M$. This means that the condition (a) holds for every $k \in M$ and $y_m \in E_{n_k}$. Using that $n_i \in N_m$ for all $i \geq m$, we obtain that $y_m \notin \overline{\bigcup_{i \geq m} E_{n_i}}$. Therefore

the set $\bigcup_{k \in M} E_{n_k}$ is discrete. Using infinite subsets M_1 and M_2 of M such that $M = M_1 \sqcup M_2$, according to Corollary 3.2, we obtain that

$$\left(\overline{\bigcup_{k \in M_1} E_{n_k}} \right) \cap \left(\overline{\bigcup_{k \in M_2} E_{n_k}} \right) = \emptyset.$$

But this contradicts to the fact that the sequence $(E_{n_k})_{k \in M}$ weakly converges to y_0 .

Now we show that the sequence $(A_k)_{k=1}^\infty$ weakly converges to every point $y \in \bigcup_{k=1}^\infty A_k$.

Let $y_m \in \bigcup_{k=1}^{\infty} A_k$. Suppose that $(A_k)_{k=1}^{\infty}$ does not weakly converge to y_m . Then there exists an infinite set $M \subseteq \mathbb{N}$ such that $y_m \notin \overline{\bigcup_{k \in M} A_k}$. Without loss of the generality we can propose that $\{n_k : k \in M\} \subseteq N_m$. Note that as in the previous reasoning the set $\bigcup_{k \in M} (E_{n_k} \setminus A_k)$ is discrete. Therefore, using Corollary 3.2 we obtain that there exists an infinite set $M_1 \subseteq M$ such that $y_m \notin \overline{\bigcup_{k \in M_1} (E_{n_k} \setminus A_k)}$. Thus, $y_m \notin \overline{\bigcup_{k \in M_1} E_{n_k}}$. This contradicts to the fact that the sequence $(E_{n_k})_{k \in M_1}$ weakly converges to y_m .

(iii) \Rightarrow (i). Let a sequence $(E_n)_{n=1}^{\infty}$ of nonempty finite pairwise disjoint sets weakly converges to a point $y_0 \in Y$. For every $n \in \mathbb{N}$ we put $G_n = \bigcup_{k \geq n} U(E_k, E_{k+1})$.

It easy to see that all sets G_n are open and everywhere dense in X . Therefore the sets $F_n = X \setminus G_n$ are nowhere dense in X . Now it is sufficient to prove that $\bigcap_{n=1}^{\infty} G_n = \emptyset$.

Assume that $x_0 \in \bigcap_{n=1}^{\infty} G_n$. Then there exists a strictly increasing sequence $(k_n)_{n=1}^{\infty}$ of integers $k_n \in \mathbb{N}$ such that $x_0 \in U(E_{k_n}, E_{k_{n+1}})$ for every $n \in \mathbb{N}$, that is $x_0(y) = 0$ for every $y \in \bigcup_{n=1}^{\infty} E_{k_n}$ and $x_0(y) = 1$ for every $y \in \bigcup_{n=1}^{\infty} E_{k_{n+1}}$. Since $(E_n)_{n=1}^{\infty}$ weakly converges to y_0 in Y , the oscillation of the function x_0 on each neighborhood V of y_0 equals to 1. But this contradicts to the continuity of x_0 at y_0 . \square

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